SOLUTION-MARCH 2022 MATH QUESTION–IZTECH

Let n be a positive integer. How many positive real numbers x satisfy the equation $\cos(nx)$=x?

Solution : For a given $n \in \mathbb{Z}^+$, and for all $x \in \mathbb{R}$, $\cos(nx)$ takes values in the interval $[-1, 1]$. So, in order to the given equation be satisfied, x should take values in $[-1, 1]$. But we are also looking for the positive real numbers, so we should have $x \in (0, 1]$. In particular, when $x = 1$, we have $\cos(n) = 1$ but since $n \in \mathbb{Z}^+$, this is not possible. So overall, desired solutions of the given equation are in the interval $(0, 1)$.

Let k be a natural number. Let us call the behaviour of $\cos(nx)$ in the interval $\left(\frac{2k\pi}{n}, \frac{2(k+1)\pi}{n}\right]$ a full-cycle. Let T denote the number of full cycles of the function in the interval $(0, 1)$. Since $\cos(nx)$ and x are continuous functions, if $T = 0$, then these functions intersect in this interval at a single point. Now, for a given n value, the full-cycles of $\cos(nx)$ in our interval could be greater than or equal to 1. Our claim is, the functions $\cos(x)$ and x intersect in the interval $(0, 1)$ at $2T$+1 many points.

1. For $T = 1$, which implies $\frac{2\pi}{n} < 1 < \frac{4\pi}{n}$, there exist some real numbers $\underline{x_1}, \overline{x_1}$ such that $\cos(n\underline{x_1}) = -1$, $\cos(n\overline{x_1})$=1 and $0 < \underline{x_1} < \overline{x_1} < 1$. Since in each of the intervals $(0, \underline{x_1}), [\underline{x_1}, \overline{x_1})$ the function $\cos(nx)$ has a half-cycle and in each half-cycle $\cos(nx)$ is monotone, there exists a single point in each interval where $\cos(nx)$ and x has the same values. At the left endpoint of the interval $[\overline{x_1}, 1]$, $\cos(n\overline{x_1})$=1> $\overline{x_1}$, and at the right endpoint $cos(n)$ <1. Our two functions are monotone in this interval so we have another unique intersection point. So overall, we have 3 intersection points and our claim is true for $T = 1$.
2. Let $T = M$, in other words $\cos(nx)$ has M many full-cycles in the interval (0,1). Then, for each $j = 1, ..., M$, there exist $\underline{x_1}, ..., \underline{x_M}$ and $\overline{x_1}, ..., \overline{x_M}$ such that $0 < \underline{x_1} < \overline{x_1} < ... < \underline{x_M} < \overline{x_M} < 1$ and $\cos(n\underline{x_j}) = -1$, $\cos(n\overline{x_j}) = 1$. Assume there exist $2M$+1 intersection points.
3. Let $T = M + 1$. In this case, in addition to the points in step 2, we have $\underline{x_{M+1}}, \overline{x_{M+1}}$ such that $\cos(n\underline{x_{M+1}})$=-1, $\cos(n\overline{x_{M+1}})$=1 and $\overline{x_M} < \underline{x_{M+1}} < \overline{x_{M+1}} < 1$. Following the same method in step 1, we can show that there exist two more intersection points of $\cos(nx)$ and x that exist in the intervals

$(\underline{x_M}, \overline{x_{M+1}})$ and $[\overline{x_{M+1}}, 1]$. So overall, we have $2M + 1 + 2 = 2(M + 1) + 1$ intersection points and thus, we are done.