SOLUTION-JANUARY 2022 MATH QUESTION–IZTECH

Let A be the subset of positive integers whose members are coprime to 6. To illustrate,

$$A = \{1, 5, 7, 11, 13, 17, 19, 23, 25, ...\}.$$

Consider the sequence of sums of reciprocals of the members of A, for instance $a_1 = 1$, $a_2 = 1 + \frac{1}{5}$, $a_3 = 1 + \frac{1}{5} + \frac{1}{7}$,...,

$$a_9 = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \frac{1}{25}.$$

In other words, a_n denotes the sum of reciprocals of the first n members of A. Prove that a_n is not an integer if $n > 1$.

SOLUTION: Let b_n denote the nth element of A. Then,

$$a_n = 1 + \frac{1}{5} + \cdots + \frac{1}{b_n}.$$

As $n > 1$, there is $k \geq 1$ such that $5^k \leq b_n < 5^{k+1}$. Note that 5^k is an element of A, but $2 \cdot 5^k, 3 \cdot 5^k, 4 \cdot 5^k$ are not in A. This implies that 5^k is the unique element in A which is less than or equal to b_n. Let L be the least common multiple of the numbers $b_1, ..., b_n$. Then, $L = 5^k \cdot u$ and u cannot be divisible by 5. By rewriting a_n as

$$a_n = \frac{\frac{L}{1} + \frac{L}{5} \cdots + \frac{L}{5^k} + \cdots + \frac{L}{b_n}}{L},$$

one sees that all the terms except $u = \frac{L}{5^k}$ are divisible by 5 in the numerator of a_n. Thus, the numerator of a_n is not divisible by 5, however its denominator is divisible by 5. This concludes that a_n is never an integer when $n > 1$.