IYTE, Department of Mathematics
PhD Preliminary Exam 1 - 2008
Complex Analysis

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

1. a) Suppose that f is an entire function and has a pole at $z = \infty$. Prove that f must be a polynomial.

   b) Let the function f be analytic in the entire complex plane and real valued on the real axis. Assume that f has positive imaginary part in the upper half plane. Prove that $f^{'}(x) > 0$ for $x \in \mathbb{R}$.

2. a) Let f be a one to one, holomorphic map of the unit disk into itself. Prove that

$$|f^{'}(z)| \leq \frac{1}{1-|z|^2}.$$

   b) Let $f(z)$ be meromorphic in $\mathbb{D}$ with the zeros $a_j, j = 1, \cdots, n$ and the poles $b_k, k = 1, \cdots, m$ in $\mathbb{D}$. Assume that $f(z)$ and $g(z)$ are continuous in $\overline{\mathbb{D}}$. If $g(z)$ is analytic in $\mathbb{D}$, then compute

$$\frac{1}{2\pi i} \int_{\partial D(0,1)} g(z) \frac{f^{'}(z)}{f(z)} dz.$$

3. a) State the reflection principle for harmonic functions.

   b) Consider the semicircular region $r < r_0$, $0 < \theta < \pi$ where u is harmonic. The function u is zero on the diameter and $u = f(\theta)$ when $r = r_0, 0 < \theta < \pi$. Derive a Poisson integral formula for the semicircular region.

   (Hint: First use the reflection principle to complete the half disk to the disk. Then, apply the Poisson integral formula to the disk. Recall that $P_r(\phi - \theta) = \dfrac{1}{2\pi} \dfrac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$ is the Poisson kernel for the disk.)

4. a) Prove that a function, which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$, reduces to a polynomial.

b) Find the linear fractional transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin and the origin into i.

5. Use the Residue Calculus to evaluate the integral $\int_0^\infty \frac{x^{1/3}}{(1+x)^2} dx$.

IYTE, Department of Mathematics
PhD Preliminary Exam 2 - 2008
Complex Analysis

1. Evaluate the integrals $\int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2} dx$, where $-\infty < t < \infty$.

2. a) Suppose that f and g are entire functions and that g never vanishes. If $|f(z)| \le |g(z)|$ for all z, then show that there is a constant C such that $f(z) = Cg(z)$.

   b) if f and g are entire functions, $|f(z)| \le |g(z)|$ for all z, and g has a zero of order k at z_0, then is it true that there is a constant D such that $f(z) = Dg(z)$? Explain.

3. a) Find the number of roots of the polynomial $z^4 - 4z^3 - 11 = 0$ which lie between $|z| = 1$ and $|z| = 2$.

   b) Find a conformal map that maps the upper half plane onto the unit disc and show that it does the mapping.

4. Let $f(z) = u(x,y) + iv(x,y)$ be holomorphic in $|z| < 1$, u and v real.

   a) Show that u and v are harmonic functions.

   b) Determine whether the functions u^2, v^2 and $u^2 - v^2$ are harmonic.

   c) Show that

   $$\int_0^{2\pi} u(re^{i\theta})^2 d\theta = \int_0^{2\pi} v(re^{i\theta})^2 d\theta, \qquad for \qquad 0 < r < 1$$

   if $u(0)^2 = v(0)^2$.

IYTE, Department of Mathematics
PhD Preliminary Exam 4 - 2009
Complex Analysis

1. Prove that the Residues Theorem is applicable to integral

$$\oint_{\partial D} \frac{e^z}{\sinh(2z)} dz$$

along the boundary of the unbounded region D:$-\pi/4 < Im\, z < \pi/2$ and evaluate the integral.

2. Find the number of roots of the equation

$$e^{z-\lambda} = z, \quad (\lambda > 1)$$

in the disk $|z| < 1$. Show that the roots are real.

3. Map the common overlap of the disks $|z-1| < 1$ and $|z-i| < 1$ conformally onto the first quadrant.

4. Find positive integer values of n, such that there is a complex differentiable function (single valued !) defined on the set $D = \{z \in C : |z| > 4\}$, whose derivative is

$$\frac{z^n}{(z-1)(z-2)(z-3)}$$

IYTE, Department of Mathematics
PhD Preliminary Exam 1 - 2008
Geometry

1. Find the arc length, the curvature and the torsion of the curve
   $\mathbf{r}(\tau) = (b\tau, a\cos\tau, a\sin\tau)$, where $a > 0$, $b > 0$ for $0 \le \tau \le 2\pi$.

2. Show that any surface of revolution can be parameterized in such a way that the first fundamental form is

$$dl^2 = du^2 + F(u)dv^2$$

3. Parametric form of a surface is given by

$$\mathbf{r}(x, y) = (3x + 3xy^2 - x^3, 3y + 3x^2y - y^3, h(x, y))$$

   where $h(x, y)$ is unknown function. Find function h so that

   a) coordinates x, y become *conformal*,

   b) the surface becomes *minimal*.

4. Proof that if the components of tensor $\mathbf{V}$ are equal to the corresponding components of tensor $\mathbf{W}$ in one particular coordinate system; that is

$$V^i_{jkl} = W^i_{jkl}$$

   then the tensor $\mathbf{V}$ is *equal* to tensor $\mathbf{W}$, or

$$\tilde{V}^i_{jkl} = \tilde{W}^i_{jkl}$$

   in *all* coordinate systems.

5. a) Compute the following external derivative

$$d\,(\sin x^2\,dx^1 + x^1x^2\,dx^3 + x^1x^2\,dx^2 \wedge dx^3)$$

   b) Let $\Omega = (x-a)\,dx + (y-b)\,dy + (z-c)\,dz$ is the one form in Euclidean space, a,b,c = const. Solve equation

$$\Omega \wedge *\Omega = 0$$

IYTE, Department of Mathematics
PhD Preliminary Exam 2 - 2008
Geometry

1. Given two form

$$A = (x-1)\,dy \wedge dz + (y-2)\,dz \wedge dx + (z-3)\,dx \wedge dy$$

show that equation

$$dA = *\,A \wedge A$$

describes a sphere in three dimensional Euclidean space. Find the center and the radius of this sphere.

2. Show that for any tensor of type (0,3) following relation is valid

$$S_{ijk} = S_{(ijk)} + S_{[ijk]} + \frac{2}{3}\left(S_{[ij]k} + S_{[kj]i}\right) + \frac{2}{3}\left(S_{(ij)k} - S_{k(ij)}\right)$$

where [ ] and ( ) mean anti-symmetrisation and symmetrisation respectively.

3. If $\mathbf{r}(l)$ is a unit speed curve, κ and τ are curvature and torsion respectively, show that the triple product

$$[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = \kappa^2 \tau.$$

Prove that $\mathbf{r}(l)$ is a plane curve if and only if $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = 0$.

4. Find the Gaussian curvature for the Klein model of the Lobachevsky geometry determined by the metric

$$dl^2 = \frac{du^2 + dv^2}{v^2}$$

on the half-plane $v > 0$ (the Poincare half-plane).

IYTE, Department of Mathematics
PhD Preliminary Exam 2 - 2008
Ordinary Differential Equation

1. Consider the differential equation

$$y'' + (\sin t)y' + \frac{y}{t^2+1} = e^t$$

Show that for every initial value, the solution of the problem exists and unique for all real t.

2. Given a system of linear equation,

$$\dot{x} = Ax,$$

for n vector x where A is a $n \times n$ constant matrix on $0 \leq t < \infty$ and all eigenvalues of matrix A have negative real parts. Show that solution of the system in the form

$$\dot{x} = (A + B(t))x,$$

goes to zero at $t \to \infty$, if $B(t)$ is $n \times n$ continuous matrix on $0 \leq t < \infty$, and $\int_0^\infty ||B(t)||\, dt < \infty$.

3. Consider the system

$$\begin{aligned} x' &= -x + 2x^2 + y^2, \\ y' &= -y + xy \end{aligned}$$

a. Find all critical points of the system. Investigate stability or asymptotically stability near the critical points and identify their types.
b. Construct a Liapunov function in the form

$$V(x, y) = ax^2 + by^2$$

where a and b are constants and use this function to determine whether the critical point (0,0) of the system is asymptotically stable or at least stable.

4. Solve the boundary value problem

$$x^2 y^{(IV)} + 4xy''' + 2y'' = 0, \quad y(1) = y'(1) = 0$$

where $y(x)$ is bounded as $x \to 0$.

IYTE, Department of Mathematics
PhD Preliminary Exam 2 - 2008
Numerical Analysis

1. The points x_j are equally spaced in $[-1, 1]$ for $n \geq 1$, so that

$$x_j = \frac{2j - n}{n}, \qquad j = 0, \dots, n.$$

   (a) By defining $\pi_{n+1}(x) = (x - x_0) \dots (x - x_n)$, show that

$$\pi_{n+1}(1 - \frac{1}{n}) = -\frac{(2n)!}{2^n n^{n+1} n!}.$$

   (b) Using Stirling's formula

$$N! \sim \sqrt{2\pi}\, N^{N+1/2}\, e^{-N}, \qquad N \to \infty,$$

   verify that

$$\pi_{n+1}(1 - \frac{1}{n}) \sim -\frac{2^{n+1/2}\, e^{-n}}{n}$$

   for large n.

2. Construct a minimax polynomial $p_3 \in P_3$ on the interval $[-1, 1]$ for the function defined by $f(x) = x^4 + x$.

3. Determine the quadrature points and weights for the weight function $w : x \to -\ln x$ on the interval $[0, 1]$, for $n = 0$ and $n = 1$.

4. Write down Euler's method for the solution of the problem

$$y' = x\, e^{-5x} - 5y, \qquad y(0) = 0,$$

   on the interval $[0, 1]$ with the step size $h = 1/N$. Denoting by y_N the resulting approximation to $y(1)$, show that $y_N \to y(1)$ as $N \to \infty$.

IYTE, Department of Mathematics
PhD Preliminary Exam 1 - 2008
Partial Differential Equations

1. Consider the equation

$$uu_x + u_y = 1$$

subject to the initial condition

$$u(x,x) = 0, \quad x \in \mathbb{R}.$$

Solve for $u(x,y)$.

2. Use the method of characteristics to solve

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 1 - x, \quad 0 < x < 1, \quad t > 0, \\
u(x,0) &= x^2(1-x), \quad 0 \le x \le 1 \\
\frac{\partial u}{\partial t}(x,0) &= 0, \quad ,0 \le x \le 1 \\
\frac{\partial u}{\partial x}(0,t) &= 0, \\
u(1,t) &= 0.
\end{aligned}$$

(Hint: First find a special time independent solution of the PDE.)

3. a) Show that if u is a solution of

$$L[u] = \Delta u + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} = -F(x,y)$$

in an open set D with $F < 0$, then u cannot attain its maximum at any point of D.

b) Solve the same problem with the condition $F \le 0$.(Hint: Show that for sufficiently large α the function $e^{\alpha x}$ satisfies $L[e^{\alpha x}] > 0$.)

4. Find a Duhamel principle for the heat equation

$$\begin{aligned}
w_t - \Delta w &= h(x,y,z,t) \quad \texttt{in} \quad D, \quad t > 0 \\
w(x,y,z,0) &= 0 \\
w &= 0 \quad \texttt{on the boundary of} \quad D
\end{aligned}$$

in terms of the solution of the problem

$$\begin{aligned} u_t - \Delta u &= 0 \quad \texttt{in} \quad D, \quad t > 0 \\ u(x,y,z,0) &= f(x,y,z) \\ u &= 0 \quad \texttt{on the boundary of} \quad D, \end{aligned}$$

where D is an open set.

5. Suppose that $u(x_1, x_2, x_3, t)$ satisfy $u_{tt} - c^2 \Delta u = 0$ in the domain and $u = f(x)$, $u_t = g(x)$ for $t = 0$. Determine u in terms of the functions $f(x)$ and $g(x)$ using the method of spherical means. Do not omit the intermediate steps.

IYTE, Department of Mathematics
PhD Preliminary Exam 2 - 2008
Partial Differential Equations

1. Consider the linear Cauchy problem

$$\begin{aligned} xu_x + yu_y &= (x+y)u, \qquad x \geq 0, y \geq 0 \\ u(1,y) &= 1, 1 < y < 2. \end{aligned}$$

   a) Sketch the characteristic projections on xy-plane. Is there a critical point?

   b) Solve the problem. Find $\lim_{t\to-\infty} u$.

2. Let

$$\begin{aligned} \Delta u^{(i)} = f(x,y), \qquad \texttt{on} \qquad \Omega \\ u^{(i)} = g^{(i)}(x,y), \qquad \texttt{on} \qquad \partial\Omega, \end{aligned}$$

   where $i = 1, 2$. Show that if $g^{(1)} < g^{(2)}$ then $u^{(1)} < u^{(2)}$ in Ω.

3. Consider the homogeneous wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad 0 < x < \ell, \qquad t > 0 \\ u(x,0) &= (1-x)^3, \qquad u_t(x,0) = (1-x)^2, \qquad 0 < x < \ell \\ \frac{\partial u}{\partial x}(0,t) &= 0, \qquad u(\ell,t) = 0, \qquad t > 0. \end{aligned}$$

   Find $u(\ell/2, \ell/2)$ by the method of characteristics.

4. Consider the 1 dimensional heat equation,

$$\begin{aligned} u_t &= \Delta u, \qquad a < x < b, t > 0 \\ u(x,0) &= (x-a)(b-x), \qquad u(a,t) = 0, \qquad u(b,t) = 0 \end{aligned}$$

   in a half strip $a < x < b, t > 0$.

   a) Without solving the problem determine where and what the maximum value of u is. Explain.

   b) Solve the problem by seperation of variables.

IYTE, Department of Mathematics
PhD Preliminary Exam 1 - 2008
Functional Analysis

1. Let X be an inner product space.

   a) Write and prove the Cauchy-Schwartz inequality for functions in X.

   b) Write and prove the triangle inequality for functions in X.

   c) Show that the strong convergence in X, $f_n \longrightarrow f$, $g_n \longrightarrow g$ implies the continuity of the inner product $\langle f_n, g_n \rangle \longrightarrow \langle f, g \rangle$.

2. Let H be a Hilbert space, $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal sequence in H and $\{c_k\}_{k=1}^{\infty}$ be a sequence of complex numbers.

   a) Show that the series $\sum_{k=1}^{\infty} c_k \varphi_k$ converges in H if and only if the series $\sum_{k=1}^{\infty} |c_k|^2$ converges in R.

   b) Show that if $\sum_{k=1}^{\infty} c_k \varphi_k$ converges to φ, then $c_k = \langle \varphi, \varphi_k \rangle$.

3. Let $C[0,1]$ be a normed space with maximum norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ defined on it.

   a) Show that the integral operator

   $$Ax(t) = \int_0^1 e^{t+\xi} x(\xi) d\xi$$

   is a bounded operator on $C[0,1]$.

   b) Show that the differential operator

   $$Tx(t) = \frac{d}{dt}x(t), \quad D(T) = \{x \in C[0,1] : x' \ \textit{continuous}\},$$

   is an unbounded operator.

4. a) Let $f : H \rightarrow R$ be a bounded linear functional, where it is known that $H = kerf \oplus (kerf)^{\perp}$. Show that $dim(kerf)^{\perp} = 1$.

   b) Show that for every bounded linear functional f on a Hilbert space H, there is exactly one $x_0 \in H$ such that

   $$f(x) = \langle x, x_0 \rangle, \quad \textit{and} \ \|f\| = \|x_0\|.$$

5. Let $T : H \to H$ be a bounded linear self-adjoint operator on a Hilbert space H.

   a) Show that $\langle Tx, x \rangle$ is real for all $x \in H$.

   b) Show that the eigenvalues (if exist) of T are real.

   c) Show that the eigenvectors of T corresponding to different eigenvalues are orthogonal.

IYTE, Department of Mathematics
PhD Preliminary Exam 2 - 2008
Functional Analysis

1. Let $C[a,b]$ be the set of all continuous real-valued functions on $[0,1]$ and consider the metric spaces

$$X_1 = C[a,b], \ \ with \ \ d(f,g) = \int_a^b |f(t) - g(t)|dt,$$

$$X_2 = C[a,b], \ \ with \ \ \rho(f,g) = (\int_a^b |f(t) - g(t)|^2 dt)^{\frac{1}{2}},$$

a) Prove that $d(f,g) \leq \sqrt{(b-a)}\rho(f,g)$, for every $f, g \in C[a,b]$.

b)Prove that if a sequence $f_n(t) \in C[a,b]$ converges in X_2, then it converges also in X_1.

c) Prove the triangle inequality for functions in X_2.

2. Consider the Hilbert space $L_2[a,b]$, where

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}dt.$$

Let $K(t,s)$ be complex-valued, continuous on $[a,b] \times [a,b]$, and the operator $A : L_2[a,b] \to L_2[a,b]$ be defined by

$$Af(t) = \int_a^b K(t,s)f(s)ds.$$

a) Write the Cauchy-Schwartz inequality for functions in $L_2[a,b]$.

b) Determine whether the operator A is bounded.

c) Find the adjoint A^* and state under what condition A is self-adjoint.

3. Consider the Hilbert space $L_2[-1,1]$ of all square integrable functions on [-1,1]. Let

$$W_o = \{f \in L_2[-1,1] : f(-t) = -f(t) \; for \; \forall t \in [-1,1]\},$$

and

$$W_e = \{f \in L_2[-1,1] : f(-t) = f(t) \; for \; \forall t \in [-1,1]\}.$$

a) Prove that W_o and W_e are closed subspaces of $L_2[-1,1]$ and that $W_o = (W_e)^{\perp}$.

b) Prove that $L_2[-1,1] = W_o \oplus W_e$.

c) Give an example of orthonormal basis for W_o, and an example of orthonormal basis for W_e.

4. **a)** Prove that a linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and onto.

b) Let $\beta = \{e_1, e_2, ..., e_k, ...\}$ be an orthonormal basis for the Hilbert space H and T be a linear operator in H defined by

$$Te_k = e_{k+1}, \quad k = 1, 2, ...$$

Is T isometric? Is T unitary?

IYTE, Department of Mathematics
PhD Preliminary Exam 1 - 2008
Numerical Analysis

1. Let $g(x)$ be continuous on $[a, b]$, and assume $g([a, b]) \subset [a, b]$. Furthermore, assume there is a constant $0 < \lambda < 1$, with

$$|g(x) - g(y)| \leq \lambda |x - y| \qquad \text{for all} \quad \text{x, y} \in \text{[a, b]} \tag{1}$$

   Then $x = g(x)$ has a unique solution $\alpha \in [a, b]$. Also, the iterates

$$x_n = g(x_{n-1}), \qquad n \geq 1 \tag{2}$$

   will converge to α for any choice of x_0 in $[a, b]$, and

$$|\alpha - x_n| \leq \frac{\lambda^n}{1 - \lambda} |x_1 - x_0| \tag{3}$$

2. Construct a natural cubic spline for the function $f(x) = \sin(\frac{\pi x}{2})$ over a set of three knots ($x_0 = -1$, $x_1 = 0$ and $x_1 = 1$) in the interval $[-1, 1]$.

3. Use a system of orthogonal polynomials to find the polynomial of degree two $p(x) = a_0 + a_1 x + a_2 x^2$, which minimize

$$\int_{-1}^{1} \frac{(\arccos x - p(x))^2}{\sqrt{1 - x^2}} \, dx \tag{4}$$

4. Consider the composite trapezium rule

$$\int_a^b f(x) \, dx \approx h \left(\frac{1}{2} f(x_0) + f(x_1) + \ldots f(x_{m-1}) + \frac{1}{2} f(x_m) \right)$$

   where $h = \frac{b-a}{m}$ and $x_i = a + ih$, $i = 0, 1, \ldots, m$.

   (a) Prove that the error $E_1(f)$ in the composite trapezium rule is given by

$$|E_1(f)| \leq \frac{1}{12m^2} (b - a)^3 M_2 \tag{5}$$

   where $M_2 = \max_{\zeta \in [a,b]} |f''(\zeta)|$

(b) Find the minimum value of m in the composite trapezium rule to calculate the integral

$$\int_{-1}^{1} \sin^2 x \, dx$$

with the accuracy $|E_1(f)| \leq 3 * 10^{-4}$

5. Consider the one-step method

$$y_{n+1} = y_n + h(ak_1 + bk_2)$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \alpha h, y_n + \beta h k_1)$$

and where a, b, α, β are real parameters. Show that there is a choice of these parameters such that the order of the method is 2. Is there a choice of the parameters for which the order exceeds 2?

END OF THE EXAM

SOME USEFUL FORMULAS:

1 Legendre Polynomials:

$$P_0(x) = 1, \quad P_1(x) = x$$

$$P_{n+1}(x) = \frac{2n+1}{n+1} x \, P_n(x) - \frac{n}{n+1} P_{n-1}(x), \qquad n \geq 1$$

2 Chebychev Polynomials:

$$T_0(x) = 1, \quad T_1(x) = x$$

$$T_{n+1}(x) = 2x \, T_n(x) - T_{n-1}(x), \qquad n \geq 1$$

IYTE, Department of Mathematics
PhD Preliminary Exam 1 - 2008
Ordinary Differential Equation

1. Consider the initial value problem :

$$u' = tu^2 - t^2,\ u(0) = 1$$

on $R : |t| \leq \alpha, |u - 1| \leq \beta$.

(a) Find the restriction on parameters α, β so that the solution of the problem exist and unique by fixed point theorem.
(b) Find the first two terms of the Picard iteration.

2. (a) Derive the solution of the non-homogenous equation

$$\dot{x} = A(t)x + B(t),\ x(t_0) = x_0$$

in terms of the fundamental solution $\Phi(t)$ satisfying $\Phi(t_0) = I$ of the homogeneous equation:

$$\dot{x} = A(t)x$$

by using the variation of constants where x is an $n-$vector and A an $n \times n$ continuous matrix on $r_1 < t < r_2$.
(b) Write the solution of the equation

$$\dot{x} = (A(t) + B(t))x,\ x(t_0) = x_0$$

as an integral equation by using the part (a).

3. (a) Show that if all the solutions of homogeneous equation

$$\dot{x} = A(t)x, \quad x(t_0) = x_0$$

is bounded then the same is true for nonhomogeneous equation

$$\dot{x} = (A(t) + B(t))x, \quad x(t_0) = x_0 \quad \text{as} \quad t \to \infty$$

where x is an $n-$vector and A an $n \times n$ continuous matrix on $r_1 < t < r_2$ provided

$$\int^{\infty} ||B(t)||dt < \infty \quad \text{and} \quad Tr(A(t)) = 0.$$

(b) Show that all solutions of the differential equation

$$u'' + (1 + \frac{2}{t^2 + 1})u = 0$$

for $t > 0$ are bounded.

4. (a) Give the definition of the stability.
(b) Show that if all solution of the equation

$$\dot{x} = A(t)x, \;\; x(t_0) = x_0$$

where x is an $n-$vector and $A(t)$ an $n \times n$ continuous matrix on $r_1 < t < r_2$ are bounded then they are stable.

5. Consider the nonlinear system of equation

$$\dot{x_1} = 8x_1 - x_2^2$$

$$\dot{x_2} = -6x_2 + 6x_1^2$$

Find all critical points of the system and determine type and stability of each of these critical points.

IYTE, Department of Mathematics
PhD Preliminary Exam 3 - 2009
Ordinary Differential Equations

1. Show that for every initial value, the following problem has a unique solution and it exists for all real t,

$$x_1' = 3x_1 + t$$
$$x_2' = x_2 + t^3 x_3$$
$$x_3' = 2tx_1 - x_2 + e^t x_3.$$

2. Consider the system of equations

$$\dot{y} = Ay, \quad y(0) = u_0,$$

and

$$\dot{x} = Ax + f(t,x), \quad x(0) = u_0,$$

where x, y are $n-$vectors and A is an $n \times n$ continuous constant matrix on $r_1 < t < r_2$ provided $||f(t,x)|| \leq t^2 e^{-t}||x||$. Show that if $\lim_{t\to\infty} y(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$.

3. Consider the second order homogenous equation

$$x'' + \frac{2}{1+t}x' = 0, \quad t \geq 0,$$

a. Find the fundamental set of solutions,

b. Check the stability and asymptotic stability of solutions,

c. Find solution of the non-homogenous initial value problem

$$x'' + \frac{2}{1+t}x' = \frac{1}{(2+t)^2}$$

with the initial conditions $x(0) = 1, x'(0) = 0$,

d. Check stability and asymptotic stability for solutions of the non-homogenous equation as $t \to \infty$.

4. Consider the boundary value problem

$$x^2y'' + \frac{1}{4}y = \lambda y, \quad y(1) = y(e) = 0,$$

a. Find eigenvalues and eigenfunctions for this boundary value problem,
b. Find normalization of the eigenfunctions in the space of continous functions $L^2[1, e]$,
c. Find the interval and the weight function for which the problem becomes self-adjoint and the eigenfunctions become orthonormal.

IYTE, Department of Mathematics
PhD Preliminary Exam 3 - 2009
Agebra II

1. Let M be a maximal submodule of a module N and let K be any submodule of N. Show that either $K \leq M$ or $M \cap K$ is a maximal submodule of K.

2. Let R be a commutative ring, M be a hollow R-module of finite length and $rM \neq M$ for some $r \in R$. Show that $r^n M = 0$ for some positive integer n.

   Hint: Use Fitting's Lemma.

3. (a) Show that $\operatorname{Soc} T \trianglelefteq T$ for every nonzero torsion abelian group T.

   (b) Show that for every torsion abelian group T an injective envelope of $\operatorname{Soc} T$ is an injective envelope for T as well.

4. Consider the following diagram induced by inclusions $T(A) \to A$ and $\mathbb{Z} \to \mathbb{Q}$, where A is any abelian group and $T(A)$ is its torsion subgroup. Prove the followings:

   (a) The rows and columns of the diagram are exact;

   (b) $T(A) \otimes \mathbb{Q} = 0$

   (c) $\operatorname{Tor}(A/T(A), \mathbb{Z}) = \operatorname{Tor}(A, \mathbb{Q}) = \operatorname{Tor}(A/T(A), \mathbb{Q}/\mathbb{Z}) = 0$

   (d) $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ker} n \circ g = \operatorname{Ker} k \circ m \cong T(A) \otimes \mathbb{Z} \cong T(A)$

$$
\begin{array}{ccccccc}
 & & & & \operatorname{Tor}(A/T(A),\mathbb{Z}) & & \\
 & & & & \downarrow & & \\
 & & & & T(A)\otimes\mathbb{Z} & \longrightarrow & T(A)\otimes\mathbb{Q} \\
 & & & & \downarrow{\scriptstyle h} & & \downarrow \\
\operatorname{Tor}(A,\mathbb{Q}) & \longrightarrow & \operatorname{Tor}(A,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{f} & A\otimes\mathbb{Z} & \xrightarrow{g} & A\otimes\mathbb{Q} \\
 & & & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
 & & \operatorname{Tor}(A/T(A),\mathbb{Q}/\mathbb{Z}) & \longrightarrow & A/T(A)\otimes\mathbb{Z} & \xrightarrow{k} & A/T(A)\otimes\mathbb{Q}
\end{array}
$$

IYTE, Department of Mathematics
PhD Preliminary Exam 3 - 2009
Complex Analysis

1. Evaluate
$$\oint_C \frac{f'(z)}{f(z)} dz$$
if C is the circle $|z| = \pi$ and $f(z) = \sin \pi z$.

2. Find number of roots for equation
$$z^8 - 4z^5 + z^2 - 1 = 0$$
in domain $|z| < 1$.

3. a) Prove that the angle between two curves C_1 and C_2 passing through the point z_0 in the z plane is preserved under the transformation $w = f(z)$, t.e. the mapping is conformal, if $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$.

   b) Find the Möbius transformation which maps the half of the plane z: $-\frac{3\pi}{4} < Argz < \frac{\pi}{4}$ into the unit circle in the w plane in such a way that $z = 1 - i$ is mapped into $w = 0$, while the point at infinity is mapped into $w = -1$.

4. Evaluate the integral
$$\int_0^\infty \frac{1 - \cos \alpha x}{x^2} dx$$
where α is an arbitrary real number.

IYTE, Department of Mathematics
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Functional Analysis

1. Consider the normed space of continuous functions $C[-1,1]$ with the norm $\|f\| = max_{t\in[-1,1]}|f(t)|$.

   a) Prove that norm $\|.\|$ is a continuous functional. Is it linear?

   b) Linear operators defined on $C[-1,1]$ are given as follows

$$T_1 f = e^{-t} f(t), \quad T_2 f = \int_{-1}^{1} e^{(t+s)} f(s) ds, \quad t \in [-1,1].$$

   Do the operators T_1 and T_2 commute? Check whether the operators T_1, T_2 and $T_1 T_2$ are bounded or not.

2. Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator.

   a) If $M_1 \subset H_1$, $M_2 \subset H_2$, where M_1, M_2 are closed subspaces, show that

$$T(M_1) \subset M_2 \quad \Longleftrightarrow \quad T^*(M_2^\perp) \subset M_1^\perp.$$

   b) Prove that $ker(T)$ and $(ker(T))^\perp$ are closed subspace of H_1.

3. Let $T : H \to H$ be a bounded linear operator on a complex Hilbert space H.

   a) Prove that if $\langle Tx, x\rangle = 0, \quad \forall x \in H$, then $T = 0$.

   b) Show that T is normal if and only if

$$\|T^* x\| = \|Tx\|, \quad \forall x \in H.$$

   c) Give an example of a normal bounded linear operator which is not self-adjoint and unitary.

4. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H.

   a) Illustrate with an example that a convergent series $\sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$ need not have a sum x.

   b) Prove that for all $x \in H$ one has the unique representation

   $x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$ if and only if $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2$.

IYTE, Department of Mathematics
PhD Preliminary Exam 3 - 2009
Geometry

1. Show that for a curve lying on a sphere of radius a and such that the torsion τ is never 0, the following equation is satisfied

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)^2 = a^2$$

2. Find the Gaussian K and the mean curvature H of the surface

$$z = e^x \sin y$$

3. Given differential form in R^2

$$A = \left(\frac{-y}{x^2+y^2} + x^2 - y^2\right) dx + \left(\frac{x}{x^2+y^2} - 2xy\right) dy$$

a) Find domain where it is defined;

b) Check if this form is the closed form;

c) Check if this form is the exact form and if yes then construct corresponding zero-form ;

d) Find circulation $\oint_C A$ where C: $x^2 + y^2 = 1$;

e) Show that $rot\,\mathbf{A} = 2\pi\delta(\mathbf{r})$.

4. Find the geodesics for

$$ds^2 = \frac{dx^2 + dy^2}{1 - x^2 - y^2}, \quad x^2 + y^2 < 1$$

Determine the scalar curvature R for this metric.

IYTE, Department of Mathematics
PhD Preliminary Exam January-2009
Numerical Analysis

1. Suppose that n is a non-negative integer and let x_i, $i = 0, 1, \ldots, n$ be distinct real numbers. Let the Lagrange interpolation polynomials $L_k \in P_n$ be defined as

$$L_k(x) = \prod_{i=0,\, i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Construct a polynomial $H_k \in P_{2n+1}$ such that

$$H_k'(x_i) = 0 \quad \text{and} \quad H_k(x_i) = \begin{cases} 1, & i = k; \\ 0, & i \neq k, \end{cases} \qquad i, k = 0, 1, \ldots, n$$

2. Prove that if $\varphi_j, j = 0, 1, \ldots,$ is a system of orthogonal polynomials on the interval (a, b) with respect to the positive, continuous and integrable weight function w on (a, b) (It should be understood that φ_j is a polynomial of exact degree j), then, for $j \geq 1$, the zeros of the polynomial φ_j are real, distinct, and lie in the interval (a, b).

3. A linear spline on the interval $[a, b]$ is expressed in terms of the basis functions as

$$s(x) = \sum_{k=0}^{m} \alpha_k \varphi_k(x),$$

where φ_k is itself a linear spline which vanishes at every knot except x_k, and $\varphi_k(x_k) = 1$. Instead of being required to interpolate the function f at the knots, the spline s is required to minimise $||f - s||_2$.

   a. Show that the coefficients α_k satisfy the system of equations

   $$\mathbf{A}\alpha = \mathbf{b},$$

   where the elements of the matrix $\mathbf{A}$ are

   $$A_{ij} = \int_0^1 \varphi_j(x)\varphi_i(x)\, dx$$

and the elements of $\mathbf{b}$ are

$$b_i = \int_0^1 f(x)\varphi_i(x)\,dx.$$

b. Now suppose that the knots are equally spaced, so that $x_k = kh$, $k = 0, 1, \ldots, m$, where $h = 1/m$, $m \geq 2$. Show that the matrix is tridiagonal, with $A_{ii} = \frac{2}{3}h$ for $i = 1, \ldots, m-1$, and determine the other nonzero elements of A.

4. Determine all values of the real parameter $b, b \neq 0$, for which the linear multistep method

$$y_{n+3} + (2b-3)(y_{n+2} - y_{n+1}) - y_n = h\,b\,(f_{n+2} + f_{n+1})$$

is zero-stable. Further show that there exist a value of b for which the order of the method is 4, and that if the method is zero-stable for some value of b, then its order cannot exceed 2.

IYTE, Department of Mathematics
PhD Preliminary Exam 3 - 2009
Partial Differential Equations

1. Let $u_1(x,t)$ and $u_2(x,t)$ denote the solutions of the equation,

$$u_t = u_{xx}, \qquad t > 0, \qquad 0 < x < L$$

with initial and boundary conditions respectively $u_1(x,0) = g_1(x)$, $u_1(0,t) = f_1(t)$, $u_1(L,t) = h_1(t)$ and $u_2(x,0) = g_2(x)$, $u_2(0,t) = f_2(t)$, $u_2(L,t) = h_2(t)$. Assume that $g_1 \le g_2$, $f_1 \le f_2$ and $h_1 \le h_2$. Prove that $u_1 \le u_2$ for $x \in [0,L]$ and $t > 0$.

2. a) Find the general solution of the PDE, $u_x + 2yu_y = 0$

   b) Impose $u(0,y) = y$. Find u explicitly.

3. Let

$$u_{xx} + u_{yy} = 0, \qquad x^2 + y^2 < 1$$
$$u = y^2, \qquad x^2 + y^2 = 1.$$

   a) Solve the problem by Fourier series.

   b) Derive the Poisson integral formula for $n = 2$ for the unit disc and then solve the above problem.

4. Consider the initial-boundary value problem

$$u_{tt} - u_{xx} = tx^2 \quad \text{for } -\infty < x < \infty \qquad \text{and} \qquad t > 0,$$
$$u(x,0) = 0, u_t(x,0) = 0 \qquad \text{for} \qquad -\infty < x < \infty.$$

Use the Duhamel's principle to solve it.